ON K-JET FIELD APPROXIMATIONS OF GEODESIC DEVIATION EQUATIONS

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ABSTRACT. Let M be a smooth manifold and $\mathcal{S}$ a spray defined on the convex cone $\mathcal{C}$ of the tangent bundle TM. It is proved that the only non-trivial k-jet approximation of the exact geodesic deviation equation of $\mathcal{S}$, linear on the deviation functions and invariant under arbitrary local coordinate transformations corresponds to the Jacobi equation. However, if linearity in the deviation functions is not required, there are differential equations whose solutions admit k-jet approximations and are invariant under arbitrary coordinate transformations. As an example of higher order geodesic deviation equations we study the first and second order jet geodesic deviation equations for a Finsler spray.

1. INTRODUCTION

Given a connection on a manifold M and two geodesics of the connection, it is a fundamental problem in mathematical physics to determine the relative motion between the pair of geodesics. This problem is of great interest in theories of gravity and in astrophysics. The solution to this problem for Lorentzian manifolds is provided by the solutions of the Jacobi equation, the Jacobi fields along Lorentzian geodesics [18, 24].

Although the geodesic equation is not linear, the Jacobi equation is a linear differential equation. Therefore, the solutions of the Jacobi equation can only describe the deviation between the geodesics under some assumptions, that sometimes are not applicable to interesting physical situations. This fact motivates the search for consistent generalizations of the Jacobi equation. Currently, there are two frameworks for generalizations of the geodesic equation extensively used in astrophysics and general relativity:

- The theory developed by Hodgkinson [12] and Ciufolini [7] and based on the *linear rapid deviation hypothesis* and whose fundamental equation we denote as the *generalized Jacobi equation* (see equation (11) below).
- The formalism developed by Bażański and others, based on equations for higher order jet fields (see for instance [2, 14]).

In this work we consider the problem of the general covariance of the first approach in the general setting of geodesics of arbitrary sprays. The methodology used involves a generalization of the second approach to arbitrary sprays.

One difficulty of the first approach is the identification of the geometric character of the solutions of the generalized Jacobi equation. It was proven that the solutions cannot be tensorial [9]. Our suggestion in this paper is to consider k-jet fields for the solutions of the generalized Jacobi equation. This is a natural choice, because

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any local, natural, smooth differential operator can be represented as a k-jet field (see Peetre's theorem, see [16], p. 176). As a result of our analysis, we conclude that for an arbitrary spray, if the solutions of the generalized geodesic deviation equation are approximated by k-jet fields, the generalized Jacobi equation from Hodgkinson [12] and [7] is not general covariant. The proof of this fact is based on the analysis of the hypothesis under which the equation is obtained, showing that under some specific coordinate transformations, such hypothesis breaks down.

The structure of this work is the following. In *section* 2, we introduce some technical definitions that will be used through the paper. In particular, we define the notions of *general covariance*, *approximation scheme*, *exact geodesic deviation equation*, *generalized geodesic deviation equation* and *jet approximations*. We state the main problem considered in this paper. Our results from the analysis of the problem is presented in the form of *Theorem A* and *Corollary B*. In *section* 3, we develop further the notion of jet approximation to deviation equations. *Section* 4 contains the proofs of the main results and also *Lemma* 4.1. In *section* 5, we discuss higher order geodesic deviation equations. The idea is applied to connections associated with a Finsler spray. We make some digression on the consequences that the higher order deviation equations have for Finslerian cosmology and Finslerian relativity. Finally, in *section* 6, we discuss the results and their relation with previous work.

2. Preliminaries and results

Let M be an n-dimensional smooth manifold and $\mathcal{C} \subset TM$ a sub-bundle of codimension zero such that each fiber $\mathcal{C}_x$ is convex cone of T_xM. Given a coordinate chart $(U, \{z^\mu : U \to \mathbb{R}^n,\ \mu = 1, ..., n\})$, coordinate indices are indicated by Greek characters, and run from 1 to $n = dim(M)$. Repeated up and down Greek indices indicates the sum over 1 to n if nothing else is stated. We will assume that all the geometric structures required in our considerations are smooth.

Geodesic equation of a spray. A second order differential equation (or semi-spray) is a smooth vector field $\mathcal{S} \in \Gamma T\mathcal{C}$ such that $\pi_* \mathcal{S}(u) = u, \forall\, u \in \mathcal{C}$, with $\pi : T\mathcal{C} \to \mathcal{C}$ being the canonical projection. By using a local frame, a semi-spray can be expressed in the form

$$\mathcal{S}(x, y) = y^\mu \frac{\partial}{\partial x^\mu} - \mathcal{S}^\mu(x, y) \frac{\partial}{\partial y^\mu}. \tag{1}$$

If the semi-spray is homogeneous, its components are homogeneous functions of degree 1 on the y coordinates and therefore $\mathcal{S}^\mu(x, y) = y^\sigma \frac{\partial \mathcal{S}^\mu(x,y)}{\partial x^\sigma} =: y^\sigma\, \Gamma^\mu_\sigma(x, y)$. Then $\mathcal{S}$ is a spray. In this case, the integral curves of the spray $\mathcal{S}$ are invariant under affine reparameterizations.

The geodesics of $\mathcal{S}$ are the projection to M of the integral curves of $\mathcal{S}$. Also, the homogeneity condition for the spray $\mathcal{S}$ implies

$$\Gamma^\mu_\nu(x, y) = y^\rho \frac{\partial \Gamma^\mu_\nu(x, y)}{\partial y^\rho}, \quad (x, y) \in T_xM, \quad \mu, \nu = 1, ..., n.$$

Using this relation, the geodesic equation of $\mathcal{S}$ can be written as

$$\ddot{z}^\mu + \Gamma^\mu_\sigma(z, \dot{z})\, \dot{z}^\sigma = 0, \quad \mu = 1, ..., n. \tag{2}$$

The functions $\Gamma^\mu_\nu(z, \dot{z})$ are the non-linear connection coefficients (see for instance [20]).

There is associated to $\mathcal{S}$ a canonical, linear, torsion-free connection ∇ of the tangent bundle $\pi_{\mathcal{C}} : T\mathcal{C} \to \mathcal{C}$ (see for example [20], *Chapter* 1). The non-zero *formal Christoffel symbols* are defined by the expression[2]

$$\Gamma^{\mu}_{\nu\rho}(x,y) := \frac{\partial \Gamma^{\mu}_{\nu}(x,y)}{\partial y^{\rho}}, \quad \mu,\nu,\rho = 1,...,n, \quad (x,y) \in T_x M. \tag{3}$$

Then the geodesic equation (2) is written as

$$\ddot{z}^{\mu} + \Gamma^{\mu}_{\nu\sigma}(z,\dot{z})\,\dot{z}^{\nu}\dot{z}^{\sigma} = 0, \quad \mu = 1,...,n. \tag{4}$$

They define an affine connection if they live directly on M, that is, if each coefficient $\Gamma^{\mu}_{\nu\rho}(x) \in \mathcal{F}^{\infty}(M,\mathbb{R})$, for each $\mu,\nu,\rho = 1,...,n$.

We consider deviation equations associated with geodesic equations only, and we will assume that the connection is symmetric, that is, the relation

$$\Gamma^{\mu}_{\nu\sigma} = \Gamma^{\mu}_{\sigma\nu}, \quad \mu,\nu,\sigma = 1,...,n \tag{5}$$

holds in any coordinate system. However, the method of analysis can be extended to more general differential equations.

Geodesic deviation equations. Let us consider two nearby geodesics $x\colon I \to M$ and $X\colon I \to M$, $I \subset \mathbb{R}$ and assume, in order to simplify the treatment, that the images $x(I) \subset M$ and $X(I) \subset M$ of both geodesics are on the chart domain $U \subset M$. Let $\{\xi^{\mu} : I \to \mathbb{R},\ \mu = 1,...,n\}$ be the coordinate displacement between the geodesics defined as

$$\xi^{\mu}(s) := x^{\mu}(s) - X^{\mu}(s), \quad s \in I. \tag{6}$$

Since $x : I \to M$ and $X : I \to M$ are solutions to the geodesic equation, we have the relation

$$\ddot{\xi}^{\mu} + \Gamma^{\mu}_{\nu\sigma}(X+\xi,\dot{X}+\dot{\xi})\left(\dot{X}^{\nu}+\dot{\xi}^{\nu}\right)\left(\dot{X}^{\sigma}+\dot{\xi}^{\sigma}\right) - \Gamma^{\mu}_{\sigma\nu}(X)\,\dot{X}^{\sigma}\,\dot{X}^{\nu} = 0. \tag{7}$$

This relation is referred to as the *exact geodesic deviation equation.* One can think the relation (7) as a constraint. Let us introduce the function $F : \mathbb{R}^{3n} \to \mathbb{R}^{n}$,

$$(\xi^{\mu},\zeta^{\mu},\vartheta^{\mu}) \mapsto \vartheta^{\mu} + \Gamma^{\mu}_{\nu\sigma}(X+\xi,\dot{X}+\zeta)\left(\dot{X}^{\nu}+\zeta^{\nu}\right)\left(\dot{X}^{\sigma}+\zeta^{\sigma}\right) - \Gamma^{\mu}_{\sigma\nu}(X)\,\dot{X}^{\sigma}\,\dot{X}^{\nu}. \tag{8}$$

Then the exact geodesic deviation equation (7) reads as $F(\xi(s),\dot{\xi}(s),\ddot{\xi}(s)) = 0$, where F is interpreted as a function along the geodesic $X : I \to M$.

Notion of general covariance. A pseudogroup of transformations of a topological space T is a collection of local homeomorphisms of T such that the composition of transformations is compatible with the collection (for a formal definition see for instance [15], p. 1 or [23]). A pseudo sub-group Γ' of a pseudogroup Γ is a subset of Γ which is a pseudogroup itself. Pseudogroups are of relevance to the problem that we are considering because the notion of *general covariance of a differential equation* can be formulated as the invariance of the differential equation under the action of certain pseudogroup of transformations.

[2] Note that some of the Christoffel symbols are chosen to be zero for any natural frame. In particular the associated with covariant derivatives along vertical directions, for instance $\nabla_{\frac{\partial}{\partial y^{\mu}}} Z = 0$ for any vector field $Z \in \Gamma T\mathcal{C}$.

Definition 2.1. Given a smooth function $G:\mathbb{R}^{n_1}\to\mathbb{R}^{n_2}$ with $m\in\mathbb{N}$, G is compatible with the pseudogroup $\Gamma^\infty(\mathbb{R}^{n_1})$ of local diffeomorphisms $\Phi:\mathbb{R}^{n_1}\to\mathbb{R}^{n_1}$ of class $\mathcal{C}^\infty$ if it is invariant by the action of the pseudogroup, i.e.,

$$G\circ\Phi = G,\quad \forall\, \Phi\in\Gamma^\infty(\mathbb{R}^{n_1}).$$

It is assumed that the variables on which the function G depends hold a representation of $\Gamma^\infty(\mathbb{R}^{n_1})$. For the case that we are investigating, the relevant pseudogroup is $\Gamma^\infty(\mathbb{R}^n)$. We consider the class of $\mathcal{C}^\infty$ coordinate transformations since we will require Taylor's expansions of functions on M of arbitrary order.

Approximations of the exact deviation equation. The relation (7) is compatible with the action of each element of the pseudogroup $\Gamma^\infty(\mathbb{R}^n)$. However, the equation (7) is non-local in M, since the definition of $\{\xi^\mu : I\to M,\, \mu=1,...,n\}$ involves for each s two points on the manifold M. This implies a geometric character which is difficult to understand. One way to avoid this difficulty is by allowing the following approximations,

- The Christoffel symbols $\Gamma^\mu_{\nu\sigma}(X+\xi,\dot X+\dot\xi)$ are approximated by Taylor's series in terms of the functions ξ^μ and
- Assuming that $\{\xi^\mu,\dot\xi^\mu\}^n_{\mu=1}$ are *infinitesimal*, i.e, the monomials $\{\xi^\mu\xi^\nu$, $\xi^\mu\dot\xi^\nu$, etc...$\}$ are dropped out in a given equation (in the case of geodesic deviations, the equation is $F=0$, with F being the function (8)).

The relation (7) under the above approximations yields to the following expression, that can be thought as an ordinary differential equation

$$\ddot\xi^\mu_1 + \frac{\partial\Gamma^\mu_{\nu\sigma}}{\partial x^\rho}(X)\,\xi^\rho_1\dot X^\nu\,\dot X^\sigma + 2\,\Gamma^\mu_{\nu\sigma}(X)\,\dot\xi^\nu_1\dot X^\sigma = 0. \tag{9}$$

This is a *local equation* in the sense that only data defined along the central geodesic $X:I\to M$ is necessary in its formulation. Also, equation (9) is linear and the set of solutions defines a finite rank vector bundle of *Jacobi fields* along $X:I\to M$. Equation (9) is a non-explicit covariant way of writing the *Jacobi equation* [18]. It is compatible with the pseudogroup $\Gamma^\infty(\mathbb{R}^n)$ and one can write it in a explicitly covariant way as

$$\nabla_{\dot X}\nabla_{\dot X}J + R(J,\dot X)\dot X = 0, \tag{10}$$

where $J(s)=\xi^\mu_1(s)\frac{\partial}{\partial x^\mu}\big|_{X(s)}\in T_{X(s)}M$ and $R(J,\dot X)$ is the Riemann type curvature endomorphism of the connection ∇ determined by $J,\dot X\in T_{X(s)}M$. Then under the above hypothesis, the solutions of the exact deviation equation (7) can be approximated by the solutions of the covariant differential equation (10), that are sections of the jet bundle $J^1_0(\mathbb{R},M)$ along the central geodesic $X:I\to M$.

If one makes the assumption that only the deviation functions $\{\xi^\mu,\ \mu=1,...,n\}$ are infinitesimal (that is, that only the monomials $\{\xi^\mu\xi^\nu,\ \mu,\nu=1,...,n\}$ are negligible), the approximation of the relation (7) for any ∇ connection yields the *generalized Jacobi equation* [12],

$$\ddot\kappa^\mu + \Gamma^\mu_{\rho\nu}\left(2\dot\kappa^\rho\dot X^\nu + \dot\kappa^\rho\dot\kappa^\nu\right) + \frac{\partial\Gamma^\mu_{\rho\nu}}{\partial x^\sigma}\kappa^\sigma\left(\dot X^\rho+\dot\kappa^\rho\right)\left(\dot X^\nu+\dot\kappa^\nu\right) = 0. \tag{11}$$

Being a non-linear equation, the unknowns κ^μ have an obscure geometric interpretation. This paper is motivated by the problem of understanding the geometric nature of (11) and in particular the issue of the *general covariance* for geodesic equations associated with arbitrary smooth sprays for general sprays. This includes the

problem for affine and Finsler sprays. The covariance problem is equivalent to the compatibility of the equation (11) with the action of the pseudogroup $\Gamma^\infty(\mathbb{R}^n)$. In practice we will investigate a weaker notion of general covariance, based on the invariance under the action of certain pseudo sub-groups of $\Gamma^\infty(\mathbb{R}^n)$.

Jet approximations. Motivated by the difficulties in determining the geometric character of (11), it is reasonable to assume a specific geometric character for the solutions of the differential equation (11). The fields $\{\kappa^\mu\}$ cannot be tensorial, since for instance the space of solutions of equation (11) cannot be a vector space (for a detailed discussion of the non-tensoriality of the fields $\{\kappa^\mu\}$ see [9]). Also, the solutions are not represented by irreducible representations of the Poincaré group as the fields described in [25]. Other natural candidates for the solutions of equation (11) are sections along the curve $X : I \to M$ of a k-jet bundle $\pi_k : J^k_0(I, M) \to M$. These are smooth maps $\Psi : I \to J^k_0(I, M)$ such that the diagram

$$\begin{tikzcd} & J^k_0(I,M) \arrow[d, "\pi_k"] \\ I \arrow[ur, "\Psi"] \arrow[r, "X"'] & M \end{tikzcd} \tag{12}$$

commutes. Some relevant fields defined along $X : I \to M$ are of this type. The most relevant example is the case when $k = 1$ and $J^1_0(I, M)$ corresponds to the tangent bundle $\pi_1 : TM \to M$ and the sections $\{\xi^\mu_1 : I \to J^1_0(I, M)\}$ along $X : I \to M$ are the Jacobi vector fields by $J(s) = \xi^\mu_1(s)\frac{\partial}{\partial x^\mu}|_{X(s)}$, solutions of the Jacobi equation (9).

In order to introduce k-jet fields along the geodesic $X : I \to M$ it is convenient to work with 1-parameter geodesic variations associated with the pair of geodesics $X, x : I \to M$. This is a map

$$\Lambda : (-\epsilon_0, \epsilon_0) \times I \to M,$$

such that for each ϵ the function $\Lambda(\epsilon, \cdot) : I \to M$ is a geodesic and such that $\Lambda(0, s) = X(s)$ and there is an $\bar{\epsilon} \in (-\epsilon_0, \epsilon_0)$ such that $\Lambda(\bar{\epsilon}, s) = x(s)$. Then the local coordinate functions $\Lambda^\mu(\epsilon, s)$ are expanded in the first variable $\epsilon \in (-\epsilon_0, \epsilon_0)$ by Taylor's theorem. As a result, we find two applications for our problem. First, it allows us to approximate the fields $\{\kappa^\mu : I \to M, \mu = 1, ..., n\}$ in terms of k-jet sections and perform an analysis order by order in ϵ of the covariance of the equation (11). Second, it allows us to define k-jet sections along $X : I \to M$ and a perturbative approach to the problem of finding deviation equations. This corresponds to a generalization of Bażański's theory (see for instance [2] or [14]).

By embedding the geodesics $X : I \to M$ and $x : I \to M$ in the ribbon $\Lambda((-\epsilon_0, \epsilon_0) \times I) \subset M$ we can approximate the functions $\{\xi^\mu, \mu = 1, ..., n\}$, solutions of the exact deviation condition (7), by using Taylor's expansion

$$\xi^\mu(s) := \Omega^\mu_k(s) + \mathcal{O}(\bar{\epsilon}^{k+1}) = \sum_{i=0}^{k} \bar{\epsilon}^i \, \frac{1}{i!}\Lambda^\mu_i(s) + \mathcal{O}(\bar{\epsilon}^{i+1}) - \Lambda^\mu(0, s),$$

for a fixed $\bar{\epsilon} \in (-\epsilon_0, \epsilon_0)$. Indeed, it is useful for a variational interpretation of the deviation functions ξ^μ to consider the following *ribbon coordinate functions*,

$$\Xi^\mu : (-\epsilon_0, \epsilon_0) \times I \to \mathbb{R}, \quad (\epsilon, s) \mapsto \Lambda^\mu(\epsilon, s) - \Lambda^\mu(0, s). \tag{13}$$

The error in the approximation

$$\Xi^\mu(\epsilon, s) \to \Omega^\mu_k(\epsilon, s)$$

is given by the remainder term of the approximation, which is of order ϵ^{k+1}. Therefore, the error in the approximation $\xi^\mu(s)$ by $\Omega^\mu_k(\bar{\epsilon}, s)$ is of order $\bar{\epsilon}^{k+1}$.

The approximation $\Xi^\mu \to \Omega^\mu_k$ is an application of Taylor's theorem. One problem considered in this work is if the further approximation

$$\{\kappa^\mu(s), \mu = 1, ..., n\} \to \{\Omega^\mu_k(\bar{\epsilon}, s)\}, \tag{14}$$

where $\{\kappa^\mu(s),\ \mu = 1, ..., n\}$ is a solution of the equation (11) and $\bar{\epsilon} \in (-\epsilon_0, \epsilon_0)$ is fixed as before, is consistent with the pseudogroup $\Gamma^\infty(\mathbb{R}^n)$. For arbitrary sprays, the answer to the problem is negative. This is a question more related with the approximations involved in obtaining the differential equation (11) than with the structure of the equation itself. Indeed, the non-covariance of the approximation is generic for deviation equations (note that the equations are not necessarily geodesic equations) obtained under the same hypothesis than the generalized Jacobi equation (11).

Let us assume that in the 1-parameter family of geodesics $\Lambda : (-\epsilon_0, \epsilon_0) \times I \to M$, the parameter $\epsilon \in (-\epsilon_0, \epsilon_0)$ is invariant by the action of each element of $\Gamma^\infty(\mathbb{R}^n)$. Although sometimes such parameter can be constructed explicitly (for instance in the case of Riemannian manifolds ϵ can be the distance between the geodesics), in general we need only to assume that it exists for the geodesic families that we consider.

Approximation schemes. Let us consider a geodesic variation $\Lambda : (-\epsilon_0, \epsilon_0) \times I \to M$ associated to the original pair of geodesics $x : I \to M$ and $X : I \to M$ by an embedding (see *Proposition* 3.1).

Definition 2.2. The monomial $\xi^\mu(s)\xi^\nu(s)$ is negligible at order k if for the Taylor expansions up to order k the monomial $\Omega^\mu_k(\bar{\epsilon}, s)\Omega^\nu_k(\bar{\epsilon}, s)$ is of order $\mathcal{O}(\bar{\epsilon}^{k+1})$.

Similarly, the definition applies to monomials of type $\dot{\xi}^\mu\xi^\nu$, $\dot{\xi}^\mu\dot{\xi}^\nu$, to $\dot{\xi}^\mu\dot{\xi}^\nu\dot{\xi}^\rho$ and other monomials. The property of being negligible is covariant, since we have required that ϵ is an invariant parameter.

The different approximations found in the literature to the exact geodesic deviation equation was the motivation for the following

Definition 2.3. An *approximation scheme* is a set of negligible monomials at order k of the free algebra generated by the monomials $\{\xi^\mu, \dot{\xi}^\nu, \ddot{\xi}^\rho,\ \mu, \nu, \rho = 1, ..., n\}$.

Example 2.4. The following three examples are considered in this work:

- *Trivial approximation scheme*, where any of the monomials
$$\{\xi^\mu\xi^\nu, \xi^\mu\dot{\xi}^\nu, \dot{\xi}^\mu\dot{\xi}^\nu,\ \mu, \nu = 1, ..., n\}$$
is not negligible.
- *Linear approximation scheme*, where all the monomials
$$\{\xi^\mu\xi^\nu, \xi^\mu\dot{\xi}^\nu, \dot{\xi}^\mu\dot{\xi}^\nu,\ \ \mu, \nu = 1, ..., n\}$$
are negligible.
- *Linear rapid deviation scheme*, where from $\{\xi^\mu\xi^\nu, \xi^\mu\dot{\xi}^\nu, \dot{\xi}^\mu\dot{\xi}^\nu,\ \mu, \nu = 1, ..., n\}$ the only negligible monomials are $\{\xi^\mu\xi^\nu,\ \mu, \nu = 1, ..., n\}$.

We will consider the approximations of the type (14) and then apply a particular approximation scheme. Note that for fixed ϵ, the field $\Omega^\mu_k(\epsilon, \cdot)$ is determined by a k-jet along $X : I \to M$. The full scheme of approximations that one makes is compiled as follows:

(i) The deviation functions $\{\xi^\mu : I \to \mathbb{R},\ \mu = 1, ..., n\}$ are substituted by the ribbon coordinate functions $\{\Xi^\mu : (-\epsilon_0, \epsilon_0) \times I \to \mathbb{R},\ \mu = 1, ..., n\}$. The justification of this fact is based on *Proposition* 3.1 of *section 3*.

(ii) The ribbon coordinate functions $\{\Xi^\mu : (-\epsilon_0, \epsilon_0) \times I \to \mathbb{R}\}$ are approximated by the Taylor expansions $\{\Omega^\mu_k : (-\epsilon_0, \epsilon_0) \times I \to \mathbb{R}\}$. This approximation can be done with arbitrary accuracy for an appropriate k, by Taylor's theorem.

(iii) One assumes a particular approximation scheme. In particular the linear rapid deviation approximation scheme is equivalent to the approximation $\kappa^\mu \to \Omega^\mu$, $\mu = 1, ..., n$ in the relation (7).

These three assumptions together with the embedding proved in *Proposition* 3.1 imply that the k-jet fields $\{\Omega^\mu_k : (-\epsilon_0, \epsilon_0) \times I \to \mathbb{R}\}$ are also an approximation up to order ϵ^{k+1} for the solutions $\{\kappa^\mu : I \to M,\ \mu = 1, ..., n\}$ of the equation (11).

Method and results. The consistency criterion to be checked against general covariance is that the error in each of the approximations introduced must be bounded by or be of the same order than the error in the approximation $\{\xi^\mu \to \Omega^\mu_k(\bar{\epsilon}, s),\ \mu = 1, ..., n\}$. Note that the value $\bar{\epsilon}$ has been fixed. However, in order to have a variational interpretation, we consider the family defined by the variable parameter $\epsilon \in (-\epsilon_0, \epsilon_0)$. If $\epsilon \leq \bar{\epsilon}$, the error will be also bounded. Therefore, we consider errors bounded or of the same order than powers of ϵ. This reformulation allows to speak of order l in ϵ and make our criteria equivalent to the following: for k-jet approximations, all the error must be of order $k+1$ in ϵ, which is the error in the approximation $\Xi^\mu(\epsilon, s)$ by $\Omega^\mu_k(\epsilon, s)$. The error in considering $\xi^\mu \xi^\nu$ negligible must be comparable to the error of considering $\Omega^\mu_k \Omega^\nu_k$ negligible. To be compatible with the Taylor's approximation Ξ^μ by Ω^μ_k, this error must be of order ϵ^{k+1}, $\Omega^\mu_k \Omega^\nu_k \simeq \mathcal{O}(\epsilon^{k+1})$. Similarly, the following conditions must hold for the monomials:

$$\Omega^\mu_k \Omega^\nu_k \simeq \mathcal{O}(\epsilon^{k+1}), \quad \dot{\Omega}^\mu_k \Omega^\nu_k \simeq \mathcal{O}(\epsilon^{k+1}), \quad \dot{\Omega}^\mu_k \dot{\Omega}^\nu_k \simeq \mathcal{O}(\epsilon^{k+1}), \quad etc... \tag{15}$$

The conditions (15) can be checked order by order in $k \in \mathbb{N}$. As a consequence of such analysis, we show that using k-jet field approximations to the solutions of equation (11) one can test the compatibility with $\Gamma^\infty(\mathbb{R}^n)$ of the approximation (15) of the exact Jacobi equation. In particular, we have the following

Theorem A. Let $\mathcal{S} \in \Gamma TC$ be a spray. If the solutions of the associated generalized deviation equation (11) are k-jet fields along the geodesic $X : I \to M$, then either:

- Equation (11) is not compatible with the pseudogroup $\Gamma^\infty(\mathbb{R}^n)$, or
- The separation between the geodesics $x, X : I \to M$ is not small.

From the proof of Theorem A in *section* 3, it follows that

Corollary B. Let $\mathcal{S} \in \Gamma TC$ be a spray. Given two geodesics $X : I \to M$ and $x : I \to M$, the only k-jet approximation to the exact deviation equation (7) linear on κ and compatible with $\Gamma^\infty(\mathbb{R}^n)$ is for $k = 1$ and corresponds to the standard Jacobi equation.

The strategy to prove these results is to consider some specific non-empty pseudo sub-group $\Gamma^\infty(\Lambda, k) \subset \Gamma^\infty(\mathbb{R}^n)$ and check which approximation schemes are compatible with $\Gamma^\infty(\Lambda, k)$ for each $k \geq 0$. The pseudogroup $\Gamma^\infty(\Lambda, k)$ is more manageable than the full pseudogroup $\Gamma^\infty(\mathbb{R}^n)$. The negative outcome of our check implies the results.

3. Jet approximations of geodesic deviations

Let us consider the spray $\mathcal{S}$ and the associated connection ∇. The following result shows that locally, two nearby enough geodesics $x : I \to M$ and $X : I \to M$ can be described as a members of a 1-parameter geodesic variation for a short time interval $\tilde{I} \subset I$, with $I = [0, a]$.

Proposition 3.1. *Let $x : I \to M$ and $X : I \to M$ be two geodesics such that $x(0)$ and $X(0)$ are connected by an extensible transverse curve γ. Then there is a 1-parameter geodesic variation $\Lambda : (-\epsilon_0, \epsilon_0) \times I \to M$ such that the central geodesic is $\Lambda(0, s) = X(s)$ and $\Lambda(\bar{\epsilon}, s) = x(s)$ for some $\bar{\epsilon} \in (-\epsilon_0, \epsilon_0)$.*

Proof. The two initial points $x(0)$ and $X(0)$ can be joined by the *connecting curve* $\gamma : [0, \bar{\epsilon}] \to M$ with $\gamma(0) = X(0)$ and $\gamma(\bar{\epsilon}) = x(0)$. Indeed, one needs to extend a bit the curve γ to include $[0, \bar{\epsilon}]$ in an open interval $(0 - \sigma, \epsilon_0)$, with $\bar{\epsilon} < \epsilon_0$. To obtain the desired geodesic variation we construct appropriate initial conditions along γ. First, the tangent vector $\dot{X}(0)$ is parallel transported along γ, defining a vector field along γ denoted by $\hat{X}'(\epsilon)$. A similar operation can be done for $\dot{x}(0)$ but along the inverted curve $-\gamma$ from $x(0)$ to $X(0)$, generating a vector field $\hat{x}'(\epsilon)$. Let us consider the linear combination of vector fields along γ,

$$\mathcal{Z}(\epsilon, 0) = \frac{1}{\bar{\epsilon}} \Big(\epsilon \hat{x}'(\epsilon) + (\bar{\epsilon} - \epsilon) \hat{X}'(\epsilon) \Big) \in T_{\gamma(\epsilon)} M, \quad \bar{\epsilon} \neq 0.$$

By Picard-Lindelöf's theorem, the set of initial values $(\gamma(\epsilon), \mathcal{Z}(\epsilon, 0))$ determines an unique geodesic $\Lambda(\epsilon, \cdot) : [0, s_{max}(\epsilon)] \subset I \to M$ for some maximal $I \ni s_{max}(\epsilon) > 0$. It is clear that

$$I \ni \hat{s}_{max} := min\{s_{\max}(\epsilon), \epsilon \in [0, \bar{\epsilon}]\} > 0,$$

by compactness. By continuity, the same is true for σ small enough and s_{max} defined as

$$s_{max} := min\{s_{\max}(\epsilon), \epsilon \in (0 - \sigma, \bar{\epsilon} - \sigma)\}.$$

Therefore, we have constructed a geodesic variation $\Lambda : (0 - \sigma, \bar{\epsilon} + \sigma) \times [0, s_{max}] \to M$ with $\Lambda(0, s) = X(s)$, $\Lambda(\bar{\epsilon}, s) = x(s)$ for $s \in [0, s_{max}]$. By a convenient reparameterization of γ, the parameter in the variation can be redefined in the interval $(-\epsilon_0, \epsilon_0)$ and still keep $\Lambda(0, s) = X(s)$ and $\Lambda(\bar{\epsilon}, s) = x(s)$ in the new parameterization of Λ with the required properties. □

It is not essential which parallel transport we use in the construction of the vector field $\mathcal{Z}$. One can use for instance the parallel transport of the connection ∇ on the total lift of $\hat{\dot{X}}(\epsilon)$ to $T\mathcal{C}$, and then push-forward the corresponding vector field along γ. Another option is to use the parallel transport of any Riemannian metric defined on M.

Corollary 3.2. *Let $x, X : I \to M$ and $\Lambda : (-\epsilon_0, \epsilon_0) \times I \to M$ as in Proposition 3.1. Then there exists a $\bar{\epsilon} \in (-\epsilon_0, \epsilon_0)$ such that the relation $\xi^\mu(s) = \Lambda^\mu(\bar{\epsilon}, s) - \Lambda^\mu(0, s)$ holds.*

We can use Taylor's expansions on ϵ of the smooth maps $\Lambda^\mu : (-\epsilon_0, \epsilon_0) \times I \to \mathbb{R}$ to approximate $\xi^\mu(s)$. In particular, by application of Taylor's theorem up to order k to the function Λ^μ it follows that

$$\Lambda^\mu(\epsilon, s) = \sum_{j=0}^{k} \frac{1}{k!} \epsilon^j \frac{\partial^j \Lambda^\mu(\epsilon, s)}{\partial \epsilon^j}\Big|_{\epsilon=0} + \frac{\epsilon^{k+1}}{(k+1)!} \frac{\partial^{k+1} \Lambda^\mu(\epsilon, s)}{\partial \epsilon^{k+1}}\Big|_{\epsilon=\hat{\epsilon}(s)}, \tag{16}$$

with $\hat{\epsilon}(s) \in (0, |\epsilon|)$. Taking derivatives respect to the parameter s in (16) one obtains

$$\dot{\Lambda}^\mu(\epsilon, s) = \sum_{j=0}^{k} \frac{1}{k!}\,\epsilon^j\,\frac{\partial^j \dot{\Lambda}^\mu(\epsilon, s)}{\partial \epsilon^j}\Big|_{\epsilon=0} + \frac{\epsilon^{k+1}}{(k+1)!}\Big(\frac{\partial^{k+1}\Lambda^\mu(\epsilon, s)}{\partial \epsilon^{k+1}}\Big|_{\epsilon=\hat{\epsilon}(s)}\Big).$$

We define the following fields $\Omega^\mu_k(\epsilon, s)$ and $\varrho^\mu_{k+1}(s)$ by the relations

(17)

$$\Omega^\mu_k(\epsilon, s) = \Lambda^\mu(\epsilon, s) - \Lambda^\mu(0, s) - \frac{\epsilon^{k+1}}{(k+1)!}\,\varrho^\mu_{k+1}(s) := \sum_{i=1}^{k} \frac{\epsilon^i}{i!}\,\xi^\mu_i(s), \quad \mu = 1, ..., n.$$

For the fixed value of ϵ such that $x(s) = \Lambda(\epsilon, s)$. The corresponding time derivative are

$$\dot{\Omega}^\mu_k(\epsilon, s) := \dot{\Lambda}^\mu(\epsilon, s) - \dot{\Lambda}^\mu(0, s) - \frac{\epsilon^{k+1}}{(k+1)!}\,\dot{\varrho}^\mu_{k+1}(s) = \sum_{i=1}^{k} \frac{\epsilon^i}{i!}\,\dot{\xi}^\mu_i(s), \quad \mu = 1, ..., n.$$

The fields $\{\xi^\mu_j(s) := \frac{\partial^j \Lambda^\mu(\epsilon,s)}{\partial \epsilon^j}\Big|_{\epsilon=0}, \mu = 1, ..., n,\ j = 1, ..., k\}$ live along the central geodesic $\Lambda(0, s) = X(s)$. Ω^μ_k are k-jet bundles along $X : I \to M$. On the other hand, the *remainder fields*

$$\varrho^\mu_{k+1}(s) := \frac{\partial^{k+1}\Lambda^\mu(\epsilon, s)}{\partial \epsilon^{k+1}}\Big|_{\epsilon=\hat{\epsilon}(s)}, \quad \mu = 1, ..., n \tag{18}$$

do not live along the geodesic $X : I \to M$. Motivated by this fact, one can define the following set of transformations,

Definition 3.3. Fixed a 1-parameter geodesic variation $\Lambda : (-\epsilon_0, \epsilon_0) \to M$, $\Gamma^\infty(\Lambda, k)$ is the collection of diffeomorphisms $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that under the action of $\Gamma^\infty(\Lambda, k)$ the functions (Ξ^μ, Ω^μ_k) transforms to $(\tilde{\Xi}^\mu, \tilde{\Omega}^\mu_k)$ with the constraint

$$\Xi^\mu(\epsilon, s) - \Omega^\mu_k(\epsilon, s) = \tilde{\Xi}^\mu(\epsilon, s) - \tilde{\Omega}^\mu_k(\epsilon, s),\ \forall\, \mu = 1, ..., n. \tag{19}$$

Proposition 3.4. *$\Gamma^\infty(\Lambda, k)$ is a non-empty pseudo sub-group of $\Gamma^\infty(\mathbb{R}^n)$ with the product composition of mappings.*

Proof. To prove that it is non-empty, let us a consider transformation $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ in the same k-jet than the identity transformation $Id : \mathbb{R}^n \to \mathbb{R}^n$, but with different l-jet for $l > k + 1$. Then the restriction of Φ to $\Lambda((-\epsilon_0, \epsilon_0) \times I) \subset U \subset M$ provides an example $\varphi \in \Gamma^\infty(\Lambda, k)$. The fact that it is a pseudo sub-group it is proved by checking that the axioms of pseudogroup hold [15]. That $\Gamma^\infty(\Lambda, k)$ is a pseudo sub-group of $\Gamma^\infty(\mathbb{R}^n)$ is direct. □

Given the above Taylor's expansions of order k and a particular approximation scheme, a second order differential equation described by an algebraic expression $G(\xi, \dot{\xi}, \ddot{\xi}) = 0$ *can be approximated* by another algebraic expression $D(\xi, \dot{\xi}, \ddot{\xi}) = 0$ by equating to zero in $G(\xi, \dot{\xi}, \ddot{\xi}) = 0$ the monomials that are negligible. Such approximation is consistent with k-jet expansions if the error $G \to D$ is bounded or of the same order than the approximation $\{\xi^\mu\} \to \{\Omega^\mu_k(\bar{\epsilon}, s)\}$: if the error $G \to D$ is not bounded by the error $\{\xi^\mu\} \to \{\Omega^\mu_k\}$, then one cannot say that Ω^μ_k is a solution to $G(\Omega^\mu_k, \dot{\Omega}^\mu_k, \ddot{\Omega}^\mu_k) = 0$, which is the k-jet approximation to deviation equations that we are interested.. This imposes a constraint on the schemes of approximations that are compatible with the pseudogroup $\Gamma^\infty(\mathbb{R}^n)$. Therefore, the problem of the compatibility of a approximation scheme with $\Gamma^\infty(\mathbb{R}^n)$ is translated to check if the error $\Delta^{\mu\nu}$ in the approximation of the monomials $\xi^\mu\xi^\nu$ by $\Omega^\mu_k(\epsilon, s)\Omega^\nu_k(\epsilon, s)$ is compatible with $\Gamma^\infty(\mathbb{R}^n)$. In practical terms, it is enough to check the compatibility with $\{\Gamma^\infty(\Lambda, k), \forall k \in \mathbb{N}\}$, for all the monomials that are negligible.

4. Proofs

In this *section* we prove *Theorem* A and *Corollary* B. We assume that all the functions and expressions are smooth. We start proving the following *Lemma*,

Lemma 4.1. *For any general covariant functional expression* $G(\xi, \dot{\xi}, \ddot{\xi}) = 0$*, the only approximation schemes compatible with* $\Gamma^\infty(\mathbb{R}^n)$ *are:*

(i) The trivial approximation scheme, where none of the monomials $\{\xi^\mu\xi^\nu,\, \xi^\mu\dot{\xi}^\nu,\, \dot{\xi}^\mu\dot{\xi}^\nu,\, \dot{\xi}^\mu\dot{\xi}^\nu\xi^\rho, ...\}$ *are negligible,*

(ii) For Taylor expansions with $k = 1$ *and in the linear approximation scheme, where all the monomials* $\{\xi^\mu\xi^\nu,\, \xi^\mu\dot{\xi}^\nu,\, \dot{\xi}^\mu\dot{\xi}^\nu,\, \dot{\xi}^\mu\dot{\xi}^\nu\xi^\rho, ...\}$ *are negligible.*

Proof. The trivial approximation scheme is clearly compatible with $\Gamma^\infty(\mathbb{R}^n)$. Let us consider the linear approximation scheme. For k=1, the monomials $(\xi^\mu\xi^\nu,\, \xi^\mu\dot{\xi}^\nu,\, \dot{\xi}^\mu\dot{\xi}^\nu, \dot{\xi}^\mu\dot{\xi}^\nu\xi^\rho)$ can be approximated by monomials $(\Omega^\mu_1\Omega^\nu_1,\, \Omega^\mu_1\dot{\Omega}^\nu_1, \dot{\Omega}^\mu_1\dot{\Omega}^\nu_1, \dot{\Omega}^\mu_1\dot{\Omega}^\nu_1\Omega^\rho)$ with an error of order ϵ^2 without imposing any restriction on the character of the functions $\Omega^\mu_1(\epsilon, s)$. Therefore, such approximation scheme is compatible with $\Gamma^\infty(\mathbb{R}^n)$. Indeed, for $k = 1$ the monomial $\xi^\mu\xi^\nu$ is negligible as the following short calculation shows,

$$\Omega^\mu_1(\epsilon, s)\Omega^\nu_1(\epsilon, s) = \epsilon^2\xi^\mu_1(s)\xi^\nu_1(s), \quad \forall\epsilon \in (-\epsilon_0, \epsilon_0).$$

This implies that the term $\Omega^\mu_1(\epsilon, s)\Omega^\nu_1(\epsilon, s)$ is of the same order in ϵ than the error in the approximation $\Xi^\mu(\epsilon, s) \to \Omega^\mu_1(\epsilon, s)$ in any coordinate system, for each $\epsilon \in (-\epsilon_0, \epsilon_0)$. Since we assume that the remainder is negligible, one can neglect the term $\Omega^\mu_1(\epsilon, s)\Omega^\nu_1(\epsilon, s)$. Similarly, for the monomial $\Omega^\mu_1(\epsilon, s)\dot{\Omega}^\nu_1(\epsilon, s)$ one has the relation

$$\Omega^\mu_1(\epsilon, s)\dot{\Omega}^\nu_1(\epsilon, s) = \epsilon\xi^\mu_1(s)\epsilon\dot{\xi}^\nu_1(s) = \epsilon^2\xi^\mu_1(s)\dot{\xi}^\nu_1(s) = \mathcal{O}(\epsilon^2).$$

If we require that for k=1 the monomial $\xi^\mu\dot{\xi}^\nu$ is not negligible, then from the above relation it follows that $\Omega^\mu_1(\epsilon, s)\dot{\Omega}^\nu_1(\epsilon, s)$ must be of order $\frac{1}{\epsilon}$. Thus, it must be a smooth function $C^{\mu\nu} : I \to \mathbb{R}$ such that

$$\Omega^\mu_1(\epsilon, s)\dot{\Omega}^\nu_1(\epsilon, s) = C^{\mu\nu}(s)\frac{1}{\epsilon} \tag{20}$$

holds. However, this requirement is not compatible with $\Gamma^\infty(\Lambda, 1)$. A simple constant scaling in local coordinates $\tilde{z}^\mu(z^\mu) = \lambda z^\mu$ (that induces an element of $\Gamma^\infty(\Lambda, 1)$ by smooth restriction to the band $\Lambda((-\epsilon_0, \epsilon) \times I))$ implies a change $\tilde{\xi}^\mu_1 = \lambda\xi^\mu_1$ (the fields $\{\xi^\mu_1,\, \mu = 1, ..., n\}$ are tensorial, since they are the components of a Jacobi vector field). If we take the constant value $\lambda = \epsilon$, it follows that also $\dot{\tilde{\xi}}^\mu_1 = \lambda\dot{\xi}^\mu_1$ and therefore

$$\tilde{\Omega}^\mu_1(\epsilon, s)\dot{\tilde{\Omega}}^\nu_1(\epsilon, s) = \epsilon^2\Omega^\mu_1(\epsilon, s)\dot{\Omega}^\nu_1(\epsilon, s) = C^{\mu\nu}(s)\epsilon^2\,\frac{1}{\epsilon} = C^{\mu\nu}(s)\epsilon.$$

This is a contradiction with the relation (20), except

(i) If $C^{\mu\nu}(s) = 0$, implying that the Jacobi field $\xi^\mu_1\frac{\partial}{\partial x^\mu}$ is zero. In this case, the condition (20) is covariant, but the associated curvature endomorphism $\mathcal{R}$ must be zero, or

(ii) The parameter $\epsilon = 1$, which is a contradiction with the general requirement that $\{\xi^\mu\xi\nu,\, \mu, \nu = 1, ..., n\}$ are negligible. To see this, let us consider ϵ to be the distance function between the curves $x(s)$ and $X(x)$, for a given Riemannian metric on M. Then it is clear that if $\epsilon = 1$ there are geometric configurations where $\xi^1\xi^1$ (for instance) can be of order 1, and therefore negligible iff ξ^1 is negligible.

A similar argument follows for other monomials.

For $k \geq 2$, the argument is analogous. Let us consider the Taylor approximations of order k for ξ^μ and the corresponding expansion of the monomial

$$\Omega^\mu_k(\epsilon,s)\Omega^\nu_k(\epsilon,s) = \Big(\sum_{j=1}^{k} \frac{1}{j!}\,\epsilon^j\,\frac{\partial^j\Lambda^\mu(\epsilon,s)}{\partial\epsilon^j}\Big|_{\epsilon=0}\Big)\Big(\sum_{j=1}^{k} \frac{1}{k!}\,\epsilon^j\,\frac{\partial^j\Lambda^\nu(\epsilon,s)}{\partial\epsilon^j}\Big|_{\epsilon=0}\Big)$$

$$= \sum_{j,l=1}^{k} \frac{1}{j!l!}\epsilon^{j+l}\Big(\frac{\partial^j\Lambda^\mu(\epsilon,s)}{\partial\epsilon^j}\Big|_{\epsilon=0}\frac{\partial^l\Lambda^\nu(\epsilon,s)}{\partial\epsilon^l}\Big|_{\epsilon=0}+\frac{\partial^l\Lambda^\mu(\epsilon,s)}{\partial\epsilon^l}\Big|_{\epsilon=0}\frac{\partial^j\Lambda^\nu(\epsilon,s)}{\partial\epsilon^j}\Big|_{\epsilon=0}\Big).$$

Each of the monomials must be negligible, which implies that they must be of the same order ϵ^{k+1} than the remainder $\varrho^\mu_{k+1}(\epsilon,s)$,

$$\epsilon^{j+l-k}\Big(\frac{\partial^j\Lambda^\mu(\epsilon,s)}{\partial\epsilon^j}\Big|_{\epsilon=0}\frac{\partial^l\Lambda^\nu(\epsilon,s)}{\partial\epsilon^l}\Big|_{\epsilon=0}+\frac{\partial^l\Lambda^\mu(\epsilon,s)}{\partial\epsilon^l}\Big|_{\epsilon=0}\frac{\partial^j\Lambda^\nu(\epsilon,s)}{\partial\epsilon^j}\Big|_{\epsilon=0}\Big)\Big|_{\epsilon=0} = \mathcal{O}(\epsilon^p),$$

for some positive p. For $j = l = 1$, this statement is not covariant, since implies the condition

$$\Big(\frac{\partial\Lambda^\mu(\epsilon,s)}{\partial\epsilon}\Big|_{\epsilon=0}\frac{\partial\Lambda^\nu(\epsilon,s)}{\partial\epsilon}\Big|_{\epsilon=0}+\frac{\partial\Lambda^\mu(\epsilon,s)}{\partial\epsilon}\Big|_{\epsilon=0}\frac{\partial\Lambda^\nu(\epsilon,s)}{\partial\epsilon}\Big|_{\epsilon=0}\Big) = \mathcal{O}(\epsilon^p) \tag{21}$$

for a positive integer p. In a similar way as for the case $k = 1$, one can prove that the condition (21) is not covariant under the action of the pseudo sub-group $\Gamma^\infty(\Lambda,k)$. Therefore, non-trivial approximations schemes such that $\xi^\mu\xi^\nu$ are negligible only could work for $k = 1$ or one of the following possibilities hold:

(i) If $\epsilon = 1$, which is a contradiction with the fact ϵ is small,
(ii) If all the fields $\{\xi^\mu_k\}$ are null. This possibility is impossible for arbitrary initial conditions on the geodesics.

□

Proof of Theorem A. For $k = 1$ it follows from *Lemma* 4.1 that the approximation scheme where all the monomials $\{\xi^\mu\xi^\mu\}$ are negligible is compatible with $\Gamma^\infty(\Lambda,k)$, but then the rest of the monomials $\xi^\mu\dot{\xi}^\nu$, etc... must also be negligible. Thus, if there are no restrictions on the curvature endomorphisms, for $k = 1$ the assumptions under which equation (11) is approximated from the relation (7) does not hold in arbitrary coordinate systems and therefore, the approximation scheme leading to the generalized Jacobi equation is not invariant under the action of $\Gamma^\infty(\Lambda,k)$. Therefore, for $k = 1$, the only approximation scheme compatible with $\Gamma^\infty(\Lambda,k)$ is the linear approximation scheme, leading to the Jacobi equation (9), that we know is compatible with the pseudo sub-group $\Gamma^\infty(\Lambda,k)$. For $k \geq 2$, it follows from *Lemma* 4.1 that the approximation scheme where $\xi^\nu\xi^\mu$ is negligible is not compatible with $\Gamma^\infty(\Lambda,k)$ except in the situations when ϵ is not small (which is a contradiction with the requirement that $\{\xi^\mu\xi^\nu,\ \mu,\nu = 1,...,n\}$ are negligible.□

Corollary B follows directly from *Theorem* A.

5. Non-linear approximation schemes and Finsler space-time higher order geodesic deviation equations

We have analyzed the possible approximation schemes where ξ^μ are infinitesimal. However, *Theorem A* leaves open the possibility for the existence of alternative approximation schemes where the monomials $\{\xi^\mu\xi^\nu,\mu,\nu = 1,...,n\}$ are not negligible. In this case, one can still use Taylor's expansion on the parameter ϵ and k-jet field approximations. The method is a direct generalization of Bażański's theory from

Riemannian setting to general sprays. If $\mathcal{S}$ is a spray defined in a cone $\mathcal{C} \hookrightarrow TM$, let us consider the expansions (17) and its derivatives and insert them in the exact deviation equation (7). The connection coefficients are also develop in the variable ϵ, obtaining a finite polynomial in ϵ. Then the relation (7) can be written as a formal series,

$$\ddot{\xi}^{\mu} + \Gamma^{\mu}_{\nu\sigma}(X+\xi, \dot{X}+\dot{\xi})\left(\dot{X}^{\nu}+\dot{\xi}^{\nu}\right)\left(\dot{X}^{\sigma}+\dot{\xi}^{\sigma}\right) - \Gamma^{\mu}_{\sigma\nu}(X)\,\dot{X}^{\sigma}\,\dot{X}^{\nu} = 0 \quad \Rightarrow$$
$$\sum_{k=1}^{\infty} \epsilon^{k}\, G_k(\Xi^{\mu}, \dot{\Xi}^{\mu}, \ddot{\Xi}^{\mu}) = 0.$$

Equating to zero each term, one obtains a hierarchy of ordinary differential equations,

$$G_k(\Xi^{\mu}, \dot{\Xi}^{\mu}, \ddot{\Xi}^{\mu}) = 0, \quad k = 1, 2, 3, ... \tag{22}$$

The equation obtained from the first order $G_1(\Xi^{\mu}, \dot{\Xi}^{\mu}, \ddot{\Xi}^{\mu}) = 0$ is the Jacobi equation of the connection associated with $\mathcal{S}$. Higher order deviation equations are obtained by equating to zero the expressions $G_k(\Xi^{\mu}, \dot{\Xi}^{\mu}, \ddot{\Xi}^{\mu}) = 0$ for $k = 2, 3, ...$ These higher order geodesic deviation equations are still valid even if $\xi^{\mu}\xi^{\nu}$ are not negligible.

Higher order geodesic deviations in Finsler geometry. The spray $\mathcal{S}$ must not necessarily be affine. In particular, one can consider *Finsler sprays* living on the slit tangent bundle $N = TM \setminus \{0\}$. Then we can adopt the following generalization of the notion from J. Beem of Finsler space-time to arbitrary signature metrics [3]:

Definition 5.1. A pseudo-Finsler structure on the slit tangent bundle N is an homogeneous of degree 2 smooth map $L : N \to \mathbb{R}$ such that the vertical hessian is not degenerate on N.

Definition 5.2. A spray $\mathcal{S} \in \Gamma TN$ is a Finsler spray if there is a Finsler function $L : N \to \mathbb{R}$ such that the integral curves of $\mathcal{S}$ are the total lift of the geodesic curves of L.

Given a Finsler space-time, there is an Ehresmann connection (see for instance [11]) determined by L, defining a decomposition

$$TN = \mathcal{H} \oplus \mathcal{V}, \tag{23}$$

where $\mathcal{V} = ker(d\pi)$. An adapted frame to the horizontal-vertical decomposition is determined by the smooth tangent basis for T_uN for each $u \in N$:

(24)
$$\left\{\frac{\delta}{\delta x^1}|_u, ..., \frac{\delta}{\delta x^n}|_u, \frac{\partial}{\partial y^1}|_u, ..., \frac{\partial}{\partial y^n}|_u\right\}, \quad \frac{\delta}{\delta x^j}|_u = \frac{\partial}{\partial x^j}|_u - N^i{}_j \frac{\partial}{\partial y^i}|_u, \quad i, j = 1, ..., n,$$

where $N^i{}_j$ are called the *non-linear connection coefficients*. Given a tangent vector $X \in T_xM$ and $u \in \pi^{-1}(x)$, there is an unique horizontal tangent vector $h(X) \in T_uN$ with $d\pi(h(X)) = X$ (horizontal lift of X).

The pull-back bundle π^*TM is the maximal subset of the cartesian product $N \times TM$ such that the diagram

$$\begin{tikzcd} \pi^*TM \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & TM \arrow[d, "\pi_0"] \\ N \arrow[r, "\pi"] & M \end{tikzcd}$$

commutes. There is a *Chern's type connection* [11] defined on π^*TM and it can be shown that there are three types of curvatures associated with the spray, the Riemannian type or hh-curvature, the hv-vertical curvature and the vertical or vv-curvature. The Riemann type curvature is defined to be the tensor field R with components

(25)
$$R^{\mu}_{\nu\rho\sigma}(x,y) = \Big(\frac{\delta}{\delta x^{\sigma}}\Gamma^{\mu}_{\nu\rho} - \frac{\delta}{\delta x^{\nu}}\Gamma^{\mu}_{\sigma\rho} - \Gamma^{\mu}_{\rho\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\mu}_{\sigma\lambda}\Gamma^{\lambda}_{\rho\nu}\Big)(x,y), \quad \mu,\nu,\rho,\sigma = 1,...,n.$$

The hv-curvature of the spray $\mathcal{S}$ is the tensor whose components are

$$P^{\mu}_{\nu\rho\sigma}(x,y) := \frac{\partial \Gamma^{\mu}_{\nu\rho}}{\partial y^{\sigma}}(x,y), \quad \mu,\nu,\rho,\sigma = 1,...,n. \tag{26}$$

By the condition of homogeneity of the function L, one can see that the hv-curvature vanishes along a geodesic, $P_{(X,\dot{X})} = 0$. In the case that the associated covariant derivative is zero along vertical directions, the vertical curvature is automatically zero. This is the case for the Chern connection, whose only non-trivial curvatures are the Riemann type curvatures and hv-curvature. Note that although the tensors R and P are linked by Bianchi identities [1], in general R does not determine the hv-curvature tensor P.

In the following, we present the first and second order geodesic deviation equations for Finsler space-times. First, let us introduce some notation. The covariant vector J_2 is defined by the expression in components $J_2 = J^{\mu}_2 \frac{\partial}{\partial x^{\mu}}$, where the components are [14]

$$J^{\mu}_2 = \xi^{\mu}_2 + \Gamma^{\mu}_{\nu\rho}\xi^{\nu}_1\xi^{\rho}_1. \tag{27}$$

The vertical lift of a tangent vector $Z \in T_xM$ to $T_{(x,y)}N$ is denoted by $Z^v = Z^{\mu}\frac{\partial}{\partial y^{\mu}}$. We denote by $\mathcal{L}_Z$ the Lie derivative along Z acting on tensor fields.

Proposition 5.3. *Let $F : N \to \mathbb{R}$ be a Finsler function and consider the expansion $\xi^{\mu}(s) = \sum_{i=1}^{2} \frac{\epsilon^i}{i!}\xi^{\mu}_i(s) + \frac{\epsilon^{k+1}}{(k+1)!}\varrho^{\mu}_{k+1}(s)$. Then*

- *The first order geodesic deviation equation of a Finsler spray is the Jacobi equation,*

$$\nabla_{\dot{X}}\nabla_{\dot{X}}\xi_1 + R_{\dot{X}}(\xi_1,\dot{X})\dot{X} = 0. \tag{28}$$

- *The second order geodesic deviation equation is the non-linear differential equation*

$$\nabla_{\dot{X}}\nabla_{\dot{X}} J_2 + R_{\dot{X}}(J_2,\dot{X})\dot{X} = \nabla_{\xi_1}R(\dot{X},\xi_1)\dot{X} - \nabla_{\dot{X}}R(\dot{X},\xi_1)\xi_1 + 4R(\dot{X},\xi_1)(\nabla_{\dot{X}}\xi_1)$$
$$+\mathcal{L}_{(\nabla_{\dot{X}}\xi_1)^v}P((\nabla_{\dot{X}}\xi_1)^v,\dot{X})\dot{X} + \mathcal{L}_{(\nabla_{\dot{X}}\xi_1)^v}R(\xi_1,\dot{X})\dot{X}. \tag{29}$$

Proof. Both equations can be obtained simultaneously by developing equation (7) in powers of ϵ. The first equation is a generalization of the Jacobi equation for Finsler space-times [11], obtained by grouping together all the terms proportional to ϵ in the expansion of the exact geodesic deviation equation (7). When this is done, one obtains the condition

$$\ddot{\xi}^{\mu}_1 + \frac{\partial \Gamma^{\mu}_{\nu\sigma}}{\partial x^{\rho}}(X,\dot{X})\,\xi^{\rho}_1\dot{X}^{\nu}\,\dot{X}^{\sigma} + 2\,\Gamma^{\mu}_{\nu\sigma}(X,\dot{X})\,\dot{X}^{\sigma}\dot{\xi}^{\nu}_1 = 0.$$

A re-arrangement of this expression one obtain equation (28). Equation (29) follows from the equality $G_2 = 0$ in front of the term ϵ^2 in the exact deviation equation.

Regrouping terms that are proportional to ϵ^2, one obtains the expression

$$\ddot{\xi}^\mu_2 + \frac{\partial \Gamma^\mu_{\nu\sigma}}{\partial x^\rho}(X,\dot{X})\,\xi^\rho_2 \dot{X}^\nu\,\dot{X}^\sigma + 2\,\Gamma^\mu_{\nu\sigma}(X,\dot{X})\,\dot{X}^\sigma\dot{\xi}^\nu_2 + 2\,\Gamma^\mu_{\nu\rho}\dot{\xi}^\nu_1\dot{\xi}^\rho_1 + 4\frac{\partial \Gamma^\mu_{\nu\sigma}}{\partial x^\rho}(X,\dot{X})\,\xi^\rho_1\dot{X}^\nu\,\dot{\xi}^\sigma$$
$$+\,\xi^\lambda\xi^\sigma\frac{\partial^2\Gamma^\mu_{\nu\rho}}{\partial x^\lambda\partial x^\sigma}\dot{X}^\nu\dot{X}^\rho + \dot{\xi}^\lambda_1\dot{\xi}^\sigma_1\frac{\partial^2\Gamma^\mu_{\nu\rho}}{\partial y^\lambda\partial y^\sigma}\dot{X}^\nu\dot{X}^\rho + 2\xi^\lambda_1\dot{\xi}^\sigma_1\frac{\partial^2\Gamma^\mu_{\nu\rho}}{\partial x^\lambda\partial y^\sigma}\dot{X}^\nu\dot{X}^\rho = 0.$$

Re-arranging the terms, the above expression is equivalent to

(30)

$$\nabla_{\dot{X}}\nabla_{\dot{X}}J_2 + R_{\dot{X}}(J_2,\dot{X})\dot{X} = \nabla_{\xi_1}R(\dot{X},\xi_1)\dot{X} - \nabla_{\dot{X}}R(\dot{X},\xi_1)\xi_1 + 4R(\dot{X},\xi_1)(\nabla_{\dot{X}}\xi_1)$$

$$- \,\dot{\xi}^\lambda_1\dot{\xi}^\sigma_1\frac{\partial^2\Gamma^\mu_{\nu\rho}}{\partial y^\lambda\partial y^\sigma}\dot{X}^\nu\dot{X}^\rho - 2\xi^\lambda_1\dot{\xi}^\sigma_1\frac{\partial^2\Gamma^\mu_{\nu\rho}}{\partial x^\lambda\partial y^\sigma}\dot{X}^\nu\dot{X}^\rho. \tag{31}$$

The first line corresponds to the second order deviation equation in the case of an affine connection and is covariant (see [14]). The second line is related with the hv-curvature and is intrinsically a non-affine contribution. The functions $\{\dot{\xi}^\mu_1, \mu = 1,...,n\}$ do not define the components of a vector field along $X : I \to M$. In order to define an associated vector field, one can consider the covariant derivative $\nabla_{\dot{X}}\xi_1$ and the functions

$$(\nabla_{\dot{X}}\xi_1)^\mu = \dot{\xi}^\mu_1 + \Gamma^\mu(X,\dot{X})_{\nu\rho}\xi^\nu_1\dot{X}^\rho. \tag{32}$$

In order to re-write the second line in expression (31) in a covariant way, one can use normal coordinate systems, where the $\Gamma^\mu_{\nu\rho}(x,y) = 0$ at a fixed point[3]. In normal coordinates and using the definition of the hv-curvature, the second line can be written tensorially as

$$\dot{\xi}^\lambda_1\dot{\xi}^\sigma_1\frac{\partial^2\Gamma^\mu_{\nu\rho}}{\partial y^\lambda\partial y^\sigma}\dot{X}^\nu\dot{X}^\rho + 2\xi^\lambda_1\dot{\xi}^\sigma_1\frac{\partial^2\Gamma^\mu_{\nu\rho}}{\partial x^\lambda\partial y^\sigma}\dot{X}^\nu\dot{X}^\rho = \mathcal{L}_{(\nabla_{\dot{X}}\xi_1)^v}P((\nabla_{\dot{X}}\xi_1)^v,\dot{X})\dot{X}$$
$$+\,\mathcal{L}_{(\xi_1)^v}R(\xi_1,\dot{X})\dot{X}.$$

This is a tensorial relation and equation (29) holds in any coordinate system. □

Riemann-flat Finsler space-times. In Finsler geometry with positive definite Finsler metrics, it is a difficult problem to find metrics of non-Riemannian or Minkowski metrics with Riemannian curvature tensor zero $R = 0$, but the hv-curvature is non-trivial (see [1]; pg: 328. The existence of singular examples of such spaces for the Chern connection follows from a result from Zhou [26]). In Lorentzian signature, one way to relax the problem is to assume that the domain where the metric L is smooth is not the slit tangent bundle N and allow singular regions where L is not smooth or is degenerate. A Finsler space-time with $R = 0$ and $P \neq 0$ will be called *Riemann-flat Finsler space-time.*

In Riemann-flat space-times the first deviation equation (28) reduces to

$$\nabla_{\dot{X}}\nabla_{\dot{X}}\xi_1 = 0, \tag{33}$$

and the second deviation equation (29) reduces to

$$\nabla_{\dot{X}}\nabla_{\dot{X}}J_2 = \mathcal{L}_{(\xi_1)^v}P((\nabla_{\dot{X}}\xi_1)^v,\dot{X})\dot{X}. \tag{34}$$

Even if $R = 0$, the equation $\nabla_{\dot{X}}\nabla_{\dot{X}}\xi_1 = 0$ is not equivalent to $\ddot{\xi}_1 = 0$, since in principle there is not a coordinate system where in a whole open neighborhood $\Gamma^\mu_{\nu\rho}(X,\dot{X}) = 0$ holds. Equations (33) and (34) have an interesting interpretation in *Finslerian cosmology* (see for instance [17]), since induces an additional bending

[3]For the existence of such normal coordinate systems, see for instance [1], *Chapter 5.* Note that at the origin, the exponential map of the Chern's connection is only $\mathcal{C}^1$. However, this is enough for our argument.

for geodesic curves. This is caused by the local anisotropy of the Finsler space-time and is encoded in both fields ξ_1 and J_2. Also, it is interesting to remark that if ordinary matter is determined by the curvature R, the curvature P must be linked with the vacuum structure of the space-time.

The *local time inversion operation* exchanges the sign of the derivatives respect to the inversion of the parameter t: for a function $f \in \mathcal{F}(I, \mathbb{R})$, the local time inversion operation maps $\frac{d}{dt}f \to -\frac{d}{dt}f$. Note that in order to define this operation it is not necessary to speak of a global time orientation. The effect of the action of the time inversion operator on the geodesic $X : (a,b) \to M$ is to reverse the geodesic $\mathcal{T}X : (a,b) \to M, X(b-t+a) = X(s)$. The induced action on the tangent vector of X is $\frac{dX^\mu}{dt} \to -\frac{dX^\mu}{ds}$. Thus, if $\xi_1(t)$ is a solution of the equation (33), $\xi_1(b-t+a)$ is also a solution of the reversed differential equation (33) along the reversed geodesic. It follows easily that for Riemann-flat space-times, the first order geodesic deviation equation is invariant,

Proposition 5.4. *For Riemann-flat spaces, the Jacobi equation* (33) *is invariant under the action of the local time inversion operation.*

The second order geodesic deviation equation of a generic Finsler space-time is not invariant under $\mathcal{T}$. For instance, this is the case if the space-time metric is non-reversible. Therefore, the tensor field

$$\mathcal{P} : \Gamma TM \times \Gamma TM \Gamma TM \to \Gamma TM, \quad (X, Y, Z) \mapsto \mathcal{L}_{(\nabla_X Y)^v} P((\nabla_X Y)^v, X)Z, \tag{35}$$

when evaluated on the geodesic $(X, \dot{X})$ is sensitive to the time inversion. This time asymmetry in the geodesic deviation can be interpreted as a geometric source of irreversible evolution.

Another striking consequence of equations (33) and (34) is related with the validity of the *Einstein equivalence principle* in Finsler geometry. To show this, let us consider the Lorentzian case. In Lorentzian geometry, there are smooth normal coordinates such that the Levi-Civita coefficients $\Gamma^\mu_{\nu\rho}(x)$ are zero at a given point $p \in M$ and the coordinate functions are smooth. By a convenient choice of the coordinate systems, one can disregard the effects of the curvature term R in a small enough space-time region, and the geodesic equation describes the dynamics of a free of force, free-falling point particle. This construction is justified by the fact that normal coordinates are smooth and by the fact that in Lorentzian geometry, at the selected point $p \in M$, the connection coefficients $\Gamma^\mu_{\nu\rho}(x)$ can be approximate as $\Gamma^\mu_{\nu\rho}(x) \approx 0$ in a small neighborhood, by continuity. This follows by the possibility of approximating the curvature $R \approx 0$ in a small enough region, again by continuity. Note that this possibility is taken by guaranteed in the formulation of the week equivalence principle. However, the equations (33) and (34) forbid in general the existence of smooth normal neighborhoods. In particular, normal coordinates are only of class $\mathcal{C}^1$. Thus, the above argument by continuity of the approximation $\Gamma^\mu_{\nu\rho}(x) \approx 0$ in a small open neighborhood is not valid for Finsler geometry. The physical consequence of this fact is that Einstein's equivalence principle does not hold for generic Finsler space-times and in particular for non-trivial Riemann-flat space-times. The obstruction that avoids the existence of smooth normal neighborhoods is that the tensor P is non-zero. Indeed, the existence of normal smooth neighborhoods implies that $P = 0$. In the positive case, those spaces are called *Berwald spaces*. The analogous concept of Berwald space for Finsler space-times is straightforward,

Definition 5.5. A Finsler space-time (M, L) is of Berwald type iff for each $p \in M$ there is a coordinate system such that the connection coefficients $\Gamma^\mu_{\nu\rho}(p)$ vanish.

We have indeed the following

Proposition 5.6. *Let (M, L) be a Finsler space-time. Then if Einstein's equivalence principle holds, then the Finsler space-time (M, L) is of Berwald type.*

Proof. If Einstein's equivalence principle holds, it is possible to define a coordinate system where the effects of gravitation are locally eliminated (a freely falling laboratory coordinate system). This is equivalent to the fact that the connection coefficients of the Chern type connection are smooth and zero at one given point. Therefore, the space must be of Berwald type. Conversely, if the Finsler space-time is of Berwald type, Einstein's equivalence principle holds automatically. □

6. Discussion

In the Lorentzian case, equation (11) has been applied extensively in astrophysics (see for example [19, 4, 5, 6] and references there) and in astrometry (see for example [8]). The affine case was investigated in [21]. Although (11) looks as being non-covariant, it was argued that the generalized Jacobi equation (11) is equivalent to an explicitly covariant ordinary differential equation. Indeed, it was written in terms of the covariant derivative of the Levi-Civita connection ∇ and its curvature tensor R (see for instance, [12, 7, 4]) in a manifestly covariant way. However, the known arguments for such equivalence between the covariant and the non-covariant form of the generalized Jacobi equation (11) contain hypotheses on the curvature R or other hypotheses whose general covariance character is quite doubtable to be valid. For instance, in [12] there is a relevant passage concerning the order of magnitude of the approximations which specially significant, in particular when some second order terms in ξ^μ are disregarded, in the partial linear approximation scheme (see [12], pg 365-366). This is in contradiction with *Lemma* (4.1)), since the hypotheses that second order terms are negligible is consistent with general covariance only for $k = 1$ and for the linear approximation scheme. In particular, for rapid deviation schemes it is not longer true that the square of the deviation functions are negligible. In [4] the authors have made the assumption that the components of the Riemannian curvature tensor are small. Without more specification, this is a non-covariant statement. In [7], the author has used the assumption that certain parallel transport between the respective points $x(s)$ and $X(s)$ of the geodesics does not depend on the curve connecting them. However, this is only true under restricted curvature conditions (in particular, for trivial holonomy connections). In view of the above criticism, we should conclude that the equivalence between the equation (11) and the covariant versions found in the literature are not valid for general Lorentzian space-times and for general connection sprays. Thus, the applicability of equation (11) as approximation to (7) is restricted to special coordinate systems or for geometries with special constraints in the curvature such that the holonomy of ∇ restricted to the chart U is trivial.

It was argued in [22] that there is not a consistent generalization of the geodesic deviation in the *rapidly deviation regime*. Schutz's analysis relies on the prescription that the geodesic curves are joined by geodesics and is restricted to Riemann normal coordinates. In contrast, we have only required an initial curve connecting the initial points $x(0)$ and $X(0)$. This method extends the conclusions of [22] to more general pairs of geodesics, not necessarily with image on normal coordinate domains. Moreover, we have seen that there are general covariant generalizations of the Jacobi equation (see for instance [2, 14] and the discussion the deviation

equations for Finsler sprays). However, one needs to abandon the requirement that the monomials $\{\xi^\mu \xi^\nu,\ \mu\nu = 1, ..., n\}$ are negligible.

The apparent contradiction between the conclusions of this paper and [22] with the use of the generalized Jacobi equation (11) in astrophysical applications can be disentangled if the generalized Jacobi equation (11) is not general covariant. Indeed, it is enough to realize that it is not compatible with the pseudo sub-group $\Gamma^\infty(\Lambda, k)$. If equation (11) is not general covariant, the arguments that have been given in [22] in normal coordinates are not necessarily applicable in Fermi coordinates, that are the coordinates where the applications of equation (11) have been developed (see for instance [4, 5, 6, 21]). Note that the *geometry of Riemann normal coordinate neighborhoods* is very different than the *geometry of Fermi coordinate neighborhoods*, and that the second type of geometry (roughly speaking, a *cigar open set*) is not as sensitive to detect rapid deviated orbits as the spherical normal coordinate geometry (roughly speaking, an *isotropic ball type open set*). Therefore, by the restriction that the geodesics stay in a tubular neighborhood around the central geodesic, the linear rapidly deviation scheme leading to (11) is consistent with the chosen geometry. Then it is applicable only if the geodesics do not deviate too much, which is in contradiction with the motivation of the equation. On the contrary, the rapid deviation approximation scheme is not consistent in a generic normal coordinate system, since the geodesics can deviate in an appreciable and fast way. This argument explains why the calculation in [22] is performed in normal coordinates and also the limitation of the use of the generalized Jacobi equation to Fermi coordinate calculations.

Equation (11) is covariant under affine coordinate transformations. Since the transformation between two Fermi coordinate systems are affine transformations, the generalized Jacobi equation (11) is covariant under local coordinate transformations from Fermi to Fermi coordinates [9]. However, as an approximation to the exact deviation equation (7), the generalized Jacobi equation (11) fails to be general covariant, since the hypothesis of the approximation scheme break down in arbitrary coordinates. This is consistent with our analysis, since the pseudo sub-group of local affine coordinate transformations is in general different than $\Gamma^\infty(\Lambda, k)$.

There are alternative frameworks to the generalized geodesic deviation for going beyond linearization in the analysis of geodesic deviations. Bażański's theory [2], is a convenient framework to investigate geodesic deviations beyond the Jacobi equation. In such formalism, one can formulate an hierarchy of general covariant differential equations for the fields $\{\xi^\mu_j, \mu = 1, ..., n, j = 1, ..., k\}$. Bażański's theory was used extensively in the investigation of geodesic motion in general relativistic space-times (see for instance in [14] and in subsequent works of those authors). We have found that such methods can be extended to connections determined by arbitrary sprays $\mathcal{S} \in \Gamma\, T\mathcal{C}$.

We have applied the generalized Bażański's theory in the case of a Finsler spray, obtaining the classical Jacobi equation in Finsler geometry (28) and the second order deviation equation (29). Such higher order geodesic deviation equations can be used if the deviation between geodesics are not negligible. In the Finslerian case and for the Chern type connection, not only the Riemannian curvature is involved, but also the *hv*-curvature tensor P is significant. Therefore, one expects that the *hv*-curvature could play a relevant role on determining the topology of the pseudo-Finsler spaces. For positive Finsler spaces, this is of relevance for the Berwald-Landsberg problem [10] and have implications for Finsler space-time cosmologies. We also have seen that generically in Finsler space-times the Einstein equivalence

principle and time inversion symmetry does not necessarily hold. These effects have been demonstrated for Riemann-flat Finsler space-times, but one expects similar effects for general Finsler space-times with $R \neq 0$.

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